Tutorial 5

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1. Example 1 on P57
   Solve

$$\partial_t v - k\partial_x^2 v = 0, x > 0, t > 0$$
$$v(x, t = 0) = \phi(x) = 1, x > 0$$
$$v(x = 0, t) = 0, t > 0$$

Solution: By the solution fomula, we have

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_0^\infty e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}dy$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-p^2}dp - \frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{+\infty} e^{-q^2}dq$$
$$= [\frac{1}{2} + \frac{1}{2}\mathcal{E}rf(\frac{x}{\sqrt{4kt}})] - [\frac{1}{2} - \frac{1}{2}\mathcal{E}rf(\frac{x}{\sqrt{4kt}})]$$
$$= \mathcal{E}rf(\frac{x}{\sqrt{4kt}})$$

2. Example 2 on P58
   Solve

$$\partial_t v - k\partial_x^2 v = 0, x > 0, t > 0$$
$$v(x, t = 0) = \phi(x) = 1, x > 0$$
$$\partial_x v(x = 0, t) = 0, t > 0$$

Solution: By the solution fomula, we have

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_0^\infty e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}}dy$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-p^2}dp + \frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{+\infty} e^{-q^2}dq$$
$$= [\frac{1}{2} + \frac{1}{2}\mathcal{E}rf(\frac{x}{\sqrt{4kt}})] + [\frac{1}{2} - \frac{1}{2}\mathcal{E}rf(\frac{x}{\sqrt{4kt}})] = 1$$

3. Using reflection method to solve the following problem

$$\partial_t^2 u - c^2\partial_x^2 u = 0, \qquad x > 0, t > 0$$
$$u(x, t = 0) = \phi(x), \partial_t u(x, t = 0) = \psi(x), x > 0$$
$$\partial_x u(x = 0, t) = 0, t > 0$$

Solution: Use the reflection method, and first consider the following Cauchy Problem:

$$\partial_t^2 v - c^2\partial_x^2 v = 0, \qquad -\infty < x < \infty, t > 0$$

$$v(x,t=0)=\phi_{even}(x), \partial_t v(x,t=0)=\psi_{even}(x), -\infty<x<\infty$$

where $\phi_{even}(x)$ and $\psi_{even}(x)$ are even extension of ϕ and ψ. Then the unique solution is given by d'Alembert formula:

$$v(x,t)=\frac{1}{2}[\phi_{even}(x+ct)+\phi_{even}(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{even}(y)dy$$

And since $\phi_{even}(x)$ and $\psi_{even}(x)$ are even, so is $v(x,t)$ for $t>0$,which implies

$$\partial_x v(x=0,t)=0, t>0$$

Set $u(x,t)=v(x,t), x>0$,then $u(x,t)$ is the unique solution of Neumann Problem on the half-line. More presicely, if $x>ct$,

$$u(x,t)=\frac{1}{2}[\phi(x+ct)+\phi(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy$$

if $0<x<ct$,

$$u(x,t)=\frac{1}{2}[\phi(x+ct)+\phi(ct-x)]+\frac{1}{2c}\Big\{\int_0^{ct-x}\psi(y)dy+\int_0^{x+ct}\psi(y)dy\Big\}.$$

4. **Use Green's Theorem to prove Theorem1 on P69:**
The unique solution of

$$\begin{cases}\partial_t^2 u-c^2\partial_x^2 u=f(x,t), -\infty<x<\infty, t>0\\ u(x,t=0)=\phi(x), -\infty<x<\infty\\ \partial_t u(x,t=0)=\psi(x), -\infty<x<\infty\end{cases}$$

is

$$u(x,t)=\frac{1}{2}[\phi(x+ct)-\phi(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy+\frac{1}{2c}\iint_\Delta f(y,s)dyds$$

where Δ is the characteristic triangle.

Proof: Green's Theorem implies that

$$\iint_\Delta f(y,s)dyds=\iint_\Delta u_{tt}-c^2u_{xx}dyds=\iint_\Delta \partial_x(-c^2u_x)-\partial_t(-u_t)dyds=\int_{\partial\Delta}-u_tdy-c^2u_xds$$

Note that $\Delta=\{(y,s):0<s<t, x-c(t-s)<y<x+c(t-s)\}$ and $\partial\Delta=L_1+L_2+L_3$ with counterclockwise direction where $L_1=\{(y,0):x-ct<y<x+ct\}, L_2=\{(y,s):0<s<t, y=x+c(t-s)\}$ and $L_3=\{(y,s):0<s<t, x-c(t-s)=y\}$. Then

$$\int_{L_1}-u_tdy-c^2u_xds=\int_{x-ct}^{x+ct}-u_t(y,0)dy=\int_{x-ct}^{x+ct}-\psi(y)dy$$

$$\int_{L_2}-u_tdy-c^2u_xds=\int_{L_2}cu_tds+cu_xdy=c\int_{L_2}udu=c(u(x,t)-u(x+ct,0))=cu(x,t)-c\phi(x+ct)$$

where we have used the facts that $dy=-cds$ on L_2 and $du=u_xdy+u_sds$.

$$\int_{L_3}-u_tdy-c^2u_xds=\int_{L_3}-cu_tds-cu_xdy=c\int_{L_3}-udu=-c(u(x-ct,0)-u(x,t))=cu(x,t)-c\phi(x-ct)$$

where we have used the facts that $dy=cds$ on L_3 and $du=u_xdy+u_sds$.

Hence we have

$$u(x,t)=\frac{1}{2}[\phi(x+ct)-\phi(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy+\frac{1}{2c}\iint_\Delta f(y,s)dyds.$$